

# The Expected Number of Distinct Consecutive Patterns in a Random Permutation

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## Abstract

Let  $\pi_n$  be a uniformly chosen random permutation on  $[n]$ . Using an analysis of the probability that two overlapping consecutive  $k$ -permutations are order isomorphic, we show that the expected number of distinct consecutive patterns in  $\pi_n$  is  $\frac{n^2}{2}(1 - o(1))$ . This exhibits the fact that random permutations pack consecutive patterns near-perfectly.

# 1 Introduction

Let  $\pi_n$  be a permutation on  $[n]$ . We say that  $\pi_n$  *contains* a permutation  $\mu_k$  of length  $k$  if there are  $k$  indices  $n_1 < n_2 < \dots < n_k$  such that  $(\pi_{n_1}, \pi_{n_2}, \dots, \pi_{n_k})$  are in the same relative order as  $(\mu_1, \mu_2, \dots, \mu_k)$ . We say that  $\pi_n$  *consecutively contains* the permutation  $\mu_k$  if there are  $k$  consecutive indices  $(m, m+1, \dots, m+k-1)$  such that  $(\pi_m, \pi_{m+1}, \dots, \pi_{m+k-1})$  are in the same relative order as  $(\mu_1, \mu_2, \dots, \mu_k)$ . Let  $\phi(\pi_n)$  be the number of distinct consecutive patterns of all lengths  $k; 1 \leq k \leq n$ , contained in  $\pi_n$ . We focus on the case where  $\pi_n$  is a *uniformly chosen random permutation* on  $[n]$ , denote the random value of  $\phi(\pi_n)$  by  $X$ , and study, in this paper, its expected value  $\mathbb{E}(X)$ .

## 1.1 Distinct Subsequences and Non-Consecutive Patterns

First we summarize results on the extremal values of  $\psi(\pi_n)$ , where  $\psi(\pi_n)$  is the number of distinct (*and not necessarily consecutive*) patterns contained in  $\pi_n$ . The identity permutation reveals that

$$\min_{\pi_n \in S_n} \psi(\pi_n) = \min_{\pi_n \in S_n} \phi(\pi_n) = n + 1,$$

since the embedded patterns are  $\emptyset, 1, 12, \dots, (12 \dots n)$ . On the other hand, motivated by a question posed by Herb Wilf at the inaugural Permutation Patterns meeting, held in Dunedin in 2003 (PP2003), several authors have studied the maximum value of  $\psi(\pi_n)$ . First we have the trivial pigeonhole bound

$$\max_{\pi_n \in S_n} \psi(\pi_n) \leq \sum_{k=1}^n \min \left( \binom{n}{k}, k! \right) \sim 2^n, \quad (1)$$

which was mirrored soon after PP2003 by Coleman [4]:

$$\max_{\pi_n \in S_n} \psi(\pi_n) \geq 2^{n-2\sqrt{n}+1} \quad (n = 2^k); \quad (2)$$

which led to

$$\left( \max_{\pi_n \in S_n} \psi(\pi_n) \right)^{1/n} \rightarrow 2.$$

A team of researchers began to see if this (surprising) bound could be improved. This led to the result in [1] that

$$\max_{\pi_n \in S_n} \psi(\pi_n) \geq 2^n \left(1 - 6\sqrt{n}2^{-\sqrt{n}/2}\right), \quad (3)$$

and thus to the conclusion that  $\max_{\pi_n \in S_n} \psi(\pi_n) \sim 2^n$ . Alison Miller improved both the upper and lower bounds (2) and (3), showing in [8] that

$$2^n - O(n^2 2^{n-\sqrt{2n}}) \leq \max_{\pi_n \in S_n} \psi(\pi_n) \leq 2^n - \Theta(n 2^{n-\sqrt{2n}}). \quad (4)$$

By extracting the constants in (4) and conducting an asymptotic analysis, Fokuoh showed in [7] that the trivial upper bound actually performs better than the one in (4) for small and not-too-small values of  $n$ , though, of course (4) does better asymptotically.

In [2] the authors studied the expected number  $\mathbb{E}(\xi(W))$  of distinct subsequences contained in the word  $W = W_n$  obtained when  $n$  letters  $s_1, \dots, s_n$  are independently generated from a  $d$ -letter alphabet – with the  $i$ th letter being “typed” with probability  $\alpha_i$  (they also covered the two-state Markov case). In the simplest case, when  $d = 2$ , it was shown in [2] (with  $\alpha_1 := \alpha$ ) that asymptotically

$$\mathbb{E}(\xi(W)) \sim k(1 + \sqrt{\alpha(1-\alpha)})^n,$$

which contains the earlier result from [6] that in the equiprobable case,  $\mathbb{E}(\xi(W)) \sim k(\frac{3}{2})^n$  for a constant  $k$ .

The fact that  $\mathbb{E}(\xi(W)) \sim A^n$  for  $A < 2$  might suggest that the same is true for  $\mathbb{E}(\psi(\pi_n))$ . But consider the following argument. Since

$$k! \gg \binom{n}{k}$$

for large  $k$ , it would seem reasonable, via a heuristic “balls in boxes” argument that most or all of the patterns of large size contained in  $\pi_n$  would be distinct. It was accordingly conjectured in [7] that

$$\mathbb{E}(\psi(\pi_n)) \sim 2^n. \quad (5)$$

While we are unable to prove that (5) holds, we show in this paper that the following is true for the number  $X$  of distinct consecutive permutations contained in a random permutation:

## Main Theorem

$$\mathbb{E}(X) = \max_{\pi_n \in S_n} (\phi(\pi_n))(1 - o(1)) = \frac{n^2}{2}(1 - o(1)).$$

## 2 Proof of Main Theorem

**Lemma 2.1.** *For any  $\pi_n \in S_n$ ,*

$$\phi(\pi_n) \leq \sum_{k=1}^n \min\{(n - k + 1), k!\} \leq \frac{n^2}{2}(1 + o(1)).$$

*Proof.* There are  $k!$  permutation patterns of length  $k$ . However, not all of these can be present unless the number of consecutive positions of length  $k$ , namely  $(n - k + 1)$ , provide “enough room” for this to occur, i.e., if  $(n - k + 1) \geq k!$  This proves the first inequality. Next note that

$$\phi(\pi_n) \leq \sum_{k=1}^n \min\{(n - k + 1), k!\} \leq \sum_{k=1}^n (n - k + 1) = \sum_{k=1}^n k \leq \frac{n^2}{2}(1 + o(1)). \quad (6)$$

This completes the proof. □

**Lemma 2.2.**

$$\sum_{k=1}^n \min\{(n - k + 1), k!\} \geq \frac{n^2}{2}(1 - o(1)).$$

*Proof.* The equation  $(n - k + 1) = k!$ , by Stirling’s approximation, holds if

$$\sqrt{2\pi k} \left(\frac{k}{e}\right)^k (1 + o(1)) + k = n + 1,$$

which is true if and only if

$$(\exp\{k \log k - k + (1/2)(\log k + \log(2\pi)) + o(1)\}) (1 + o(1)) = \exp\{\log n(1 + o(1))\}.$$

A good approximation to the above is the solution to  $k \log k = \log n$ , namely

$$k = \frac{\log n}{\log \log n}(1 + o(1)) := a_n,$$

( $a_n$  will be critical in what follows), which yields

$$\sum_{k=1}^n \min\{(n-k+1), k!\} \geq \sum_{k=a_n}^n (n-k+1) \sim \frac{(n-a_n)^2}{2} = \frac{n^2}{2}(1-o(1)),$$

as asserted. □

We mention that the evidence in support of the Main Theorem is strong, as evidenced by the following data for small  $n$  (the evidence in support of (5) is not as strong; see [7])

$n$	$\sum_{k=1}^n \min(n-k+1, k!)$	Bound attained (Y/N)	$\mathbb{E}(X)$
3	4	Yes	3.67
4	6	Yes	5.83
5	9	Yes	8.7
6	13	Yes	12.33
7	18	Yes	16.78
8	24	Yes	22.08

For example, the permutation 14325 contains the  $9 = \sum_{k=1}^5 \min\{k!, 6-k\}$  patterns 1, 12, 21, 132, 321, 213, 1432, 3214, and 14325.

A study of patterns that occur in consecutive positions in a permutation is not new. For example, the so called vincular patterns partially follow this scheme. Far more relevant to this paper, however, is the work of [3] and [5] on non-consecutive permutations that touches on some of the aspects of this paper.

## 2.1 Auxiliary Random Variables

We will attack our problem using several auxiliary random variables to analyze the behavior of  $X$ . First we define  $X_k$  to be the number of distinct consecutive permutations of length  $k$  contained in a random  $n$ -permutation. We note that for some  $k_0 \geq a_n$  to be chosen later,

$$\mathbb{E}(X) = \sum_{k=1}^n \mathbb{E}(X_k) \geq \sum_{k=k_0}^n \mathbb{E}(X_k) = \sum_{k=k_0}^n ((n-k+1) - \mathbb{E}(Y_k)), \quad (7)$$

where  $Y_k$  denotes the number of  $k$  permutations that are consecutively contained in the random  $n$ -permutation as “repeats”.

The variables  $Y_k$  are difficult to work with directly, so we turn to yet another variable

$$Z_k = \sum_{j=1}^{n-k+1} \sum_{l=0}^{k-1} I_{j,l,k}, \quad (8)$$

where for  $l \geq 1$ , the indication variable  $I_{j,l,k}$  equals one iff  $(\pi_j, \dots, \pi_{j+k-1})$  and  $(\pi_{j+k-l}, \dots, \pi_{j+2k-l-1})$  are order isomorphic ( $I_{j,l,k}$  equals zero otherwise.) For  $l = 0$ ,  $I_{j,l,k} = 1$  if two non-overlapping sets of consecutive indices have the same pattern. See Figure 1. In other words  $I_{j,l,k}$  equals one iff two consecutive sets of  $k$  positions along the random  $n$ -permutation, overlapping in  $l$  positions, are order isomorphic. For two permutations  $\eta_1$  and  $\eta_2$  that are order isomorphic, we write  $\eta_1 \simeq \eta_2$ . Next, we set

$$Z = \sum_{k=k_0}^n Z_k = \sum_{k=k_0}^n \sum_{j=1}^{n-k+1} \sum_{l=0}^{k-1} I_{j,l,k}, \quad (9)$$

and make the crucial observation that

$$\{Y \geq 1\} \subseteq \{Z \geq 1\}, \quad (10)$$

where

$$Y = \sum_{k=k_0}^n Y_k, \quad (11)$$

so that

$$\mathbb{P}(Y \geq 1) \leq \mathbb{P}(Z \geq 1) \leq \mathbb{E}(Z). \quad (12)$$

Next notice that (by (12))

$$\begin{aligned} \mathbb{E}(Y) &= \sum_{j=1}^{n^2/2} j \mathbb{P}(Y = j) \\ &\leq \frac{n^2}{2} \sum_{j \geq 1} \mathbb{P}(Y = j) \\ &= \frac{n^2}{2} \mathbb{P}(Y \geq 1) \\ &\leq \frac{n^2}{2} \mathbb{E}(Z). \end{aligned} \quad (13)$$



in order for  $\eta_1$  and  $\eta_2$  to be *consistent with being order isomorphic*. Thus

$$\begin{aligned}
\mathbb{P}(\eta_1 \simeq \eta_2) &\leq \mathbb{P}(\eta_1, \eta_2 \text{ are consistent}) \\
&\leq \mathbb{P}(\rho_1 \simeq \rho_2 \simeq \rho_3, w_1 \simeq w_2) \\
&\leq \frac{1}{l!^2} \frac{1}{(k-2l)!} \\
&\leq \frac{3^k}{k!},
\end{aligned} \tag{15}$$

where the last line of (15) follows from the fact that

$$\frac{k!}{(l!^2)(k-2l)!} \leq \frac{k!}{(k/3)!^3} \sim K \cdot \frac{3^k}{k}$$

for some constant  $K$ , by Stirling's approximation. Note that the bound in (15) is uniform, i.e., independent of the value of  $l \leq k/2$ , but perhaps more importantly, it is obtained without fussing about which values of integers might actually occupy the  $2k - l$  positions of  $\eta_1 \cup \eta_2$ . An analysis that analyzes these values gets very complicated very rapidly and so we will settle for the upper bound in (15). The same kind of analysis, in which we use consistency with order isomorphism as a driving method, is used for  $l > \frac{k}{2}$ , which we turn to next.  $\square$

**Lemma 2.6.** *For  $k - 2 \geq l > \frac{k}{2}$ ,*

$$\mathbb{P}(I_{j,l,k} = 1) \leq \left( \frac{1}{(k-l)!} \right)^{\frac{k}{k-l} - 1}. \tag{16}$$

*Proof.* We first illustrate the idea of the proof for  $k = 8; l = 6$ ; see Figure 2. If the first two elements of the permutation  $\eta_1$  form the pattern  $AB$ , then so must the first two elements of  $\eta_2$ , which are also the third and fourth elements of  $\eta_1$  – forcing the third and fourth elements of  $\eta_2$  to form an  $AB$  pattern too. This repetition of the  $AB$  pattern persists till we reach the end of  $\eta_2$ , for a total of four induced  $AB$  patterns caused by the first two elements of  $\eta_1$ . Thus

$$\mathbb{P}(I_{j,k,l} = 1) \leq \left( \frac{1}{2!} \right)^4.$$

In general the pattern in the first  $k - l$  positions of  $\eta_1$  is repeated  $\lfloor \frac{k}{k-l} \rfloor \geq \frac{k}{k-l} - 1$  times, each of which has a probability  $\frac{1}{(k-l)!}$ . This completes the proof.  $\square$



$$\begin{aligned} \eta_1: & A \ B \ A \ B \ A \ B \ A \ B \\ \eta_1: & A \ B \ A \ B \ A \ B \ A \ B \end{aligned}$$

Figure 2: Consecutive Overlapping Permutations,  $l \geq \frac{k}{2}$

## 2.2 Putting it all Together

For  $k \geq k_0$  ( $k_0$  is still to be specified), we seek to find  $\sum_{l=0}^{k-1} \mathbb{P}(I_{j,l,k} = 1)$ . We address the case of  $l = 0, l = k - 1$  first. By Lemmas 2.3 and 2.4, we have

$$\mathbb{P}(I_{j,0,k} = 1) \leq \frac{1}{k!}, \quad (17)$$

and

$$\mathbb{P}(I_{j,k-1,k} = 1) \leq \frac{2}{(k+1)!}. \quad (18)$$

Now, since there are  $\leq n$  consecutive positions disjoint from the  $j$ th set, we see that

$$\sum_{k=k_0}^n \sum_j \mathbb{P}(I_{j,0,k} = 1) \leq \frac{n^3}{k!}, \quad (19)$$

and

$$\sum_{k=k_0}^n \sum_j \mathbb{P}(I_{j,k-1,k} = 1) \leq \frac{2n^2}{(k+1)!}. \quad (20)$$

For the case of small overlaps, Lemma 2.5 gives

$$\sum_{k=k_0}^n \sum_j \sum_{l=1}^{k/2} \mathbb{P}(I_{j,k,l} = 1) \leq \frac{n^2 k 3^k}{k!}. \quad (21)$$

Finally, for the large overlap case, we have

$$\begin{aligned} & \sum_{k=k_0}^n \sum_j \sum_{l=(k/2)}^{k-2} \mathbb{P}(I_{j,k,l} = 1) \\ & \leq n^2 \sum_l \left( \frac{1}{(k-l)!} \right)^{\frac{k}{k-l}-1} = n^2 \sum_l \left( \frac{1}{(k-l)!} \right)^{\frac{l}{k-l}} \\ & = n^2 \left\{ \left( \frac{1}{2!} \right)^{(k-2)/2} + \left( \frac{1}{3!} \right)^{(k-3)/3} + \dots + \left( \frac{1}{(k/2)!} \right) \right\}. \quad (22) \end{aligned}$$

In (22), the  $l = k - 2$  term is  $2(1/2)^{k/2}$ , which we treat separately. For the other terms we use the inequality  $r! \geq \sqrt{2\pi r}(r/e)^r$  to provide the estimate

$$\left(\frac{1}{(k-l)!}\right)^{\frac{l}{k-l}} \leq \left\{ \left(\frac{1}{(k-l)!}\right)^{\frac{1}{k-l}} \right\}^{k/2} \leq \left(\frac{e(1+o(1))}{k-l}\right)^{k/2} \leq (0.96)^k. \quad (23)$$

Notice that we could have been much more precise in dealing with the large overlap case. Recognizing that (23) yields the largest upper bound of the terms in equations (19) to (23), we conclude from (9) that

$$\mathbb{E}(Z) \leq 6n^3(0.96)^k, \quad (24)$$

which via (13) gives

$$\mathbb{E}(Y) \leq 3n^5(0.96)^k, \quad (25)$$

so that by (7) we have that

$$\mathbb{E}(X) \geq \left( \sum_{k=k_0}^n (n-k+1) \right) - 3n^5(0.96)^k. \quad (26)$$

We are finally ready to choose  $k_0$ , and do this in a way so that, e.g.,

$$3n^5(0.96)^k \leq n^6(0.96)^k \leq \frac{1}{n^2}. \quad (27)$$

Elementary algebra shows that (27) surely holds whenever

$$k \geq 200 \ln n := k_0, \quad (28)$$

so that

$$\begin{aligned} \mathbb{E}(X) &\geq \left( \sum_{k=200 \ln n}^n (n-k+1) \right) - \frac{1}{n^2} = \frac{(n-200 \ln n)^2}{2}(1-o(1)) - \frac{1}{n^2} \\ &= \frac{n^2}{2}(1-o(1)), \end{aligned}$$

proving the main theorem.

### 3 The Polynomial Method of AA, VD, SP, CS, and LY

Five of the authors of this paper, all REU students in the summer of 2020, are working on the final details of an alternative proof of the Main Theorem. Their paper will appear elsewhere, but the key idea is to enumerate the number of good permutations on  $[k]$ , which are polynomial of degree  $k - l$ .

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