Enumerating "Good" Permutations

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January 22, 2021

Subpermutations

Definition (Subpermutation)

Suppose π and μ are permutations. Then π is a ***subpermutation*** of μ if μ has a subsequence whose terms are equivalent to π. (Need not be contiguous)

Definition (Reduction)

Suppose that π is a subpermutation of μ. Then a ***reduction***, π_{red}, of π is the permutation obtained from replacing the ith smallest number with i.

Example

Suppose $\mu = 2\ 3\ 4\ 5\ 1$ and $\pi = 3\ 5\ 1$. Then $\pi_{red} = 2\ 3\ 1$.

Permutation Patterns

Definition (Permutation pattern)

If π is a permutation, then a ***permutation pattern*** of π is the reduction of a subpermutation of π

Example

Suppose $\pi = 2\ 3\ 4\ 5\ 1$. Then examples of permutations patterns of π are:

- 1
- 1 2
- 2 1
- 1 2 3
- 2 3 1

Order isomorphism

Definition (Order isomorphism)

Let $A = \{1, 2, ..., n\}$. Let $p(A)$ denote a permutation of A. Let π and μ be permutations of $p(A)$. We say that π is ***order isomorphic*** to μ if and only if

$$\forall \pi_i \in \pi, \pi_i < \pi_{i+1} \iff \mu_i < \mu_{i+1}.$$

Example

Suppose $\pi = \{473\}$ and $\mu = \{362\}$ are permutations. Here the elements of π and μ both have a "231" pattern, meaning that the orderings of π and μ are order isomorphic to each other.

Overlapping permutations

Definition (Overlapping permutations)

Given $n, k \in \mathbb{N}, k \leq n$, then two permutations π_{sub} and μ_{sub}, which are subpermutations on $[k]$ of two permutations π and μ on $[n]$, ***overlap*** if at least one entry i in π_{sub} and one entry j in μ_{sub} overlap in π and μ.

Example

1. Permutations can be disjoint
2. Permutations can overlap consecutively
3. Permutations can overlap non-consecutively

Good permutations

Definition (Good permutation)

If π is a permutation on $[k]$, then π is a ***good permutation*** if the first ℓ entries are order isomorphic to the last ℓ entries

Example

Suppose $\pi = 3\ 1\ 4\ 2\ 5$ and $\ell = 3$. π is a good permutation since the first $\ell = 3$ entries, $\ell_1 = 3\ 1\ 4$, and the last ℓ entries, $\ell_2 = 4\ 2\ 5$, are order isomorphic, $\ell_{1_{red}} = 2\ 1\ 3 = \ell_{2_{red}}$.

Consecutive overlap

Note

If π is a good permutation, then the order isomorphism between the first ℓ entries and the last ℓ entries of π can be seen via a consecutive overlapping permutation.

Example

If $\pi = \alpha\ \beta\ \gamma\ \delta\ \epsilon$ is a good permutation such that the first 2 entries are order isomorphic to the last 2 entries, then we can see the overlap as follows:

α	β	γ	δ	ϵ			
			α	β	γ	δ	ϵ

Sanity check

Recall and note

- There are various forms of overlapping permutations.
- We are going to focus on good permutations through consecutive overlaps.
- The number of entries in a permutation which overlap varies.
- For a permutation on $[k]$, we partition overlaps for when $\ell \leq k/2$ and for when $\ell > k/2$.

Proposition (A,D,P-C,S,Y)

The number of good permutations for a given $k \in \mathbb{N}$ and $\ell \leq k/2$ is $\binom{k}{\ell}\ell!\binom{k-\ell}{\ell}(k-2\ell)!$

Proof.

This is the result of a simple combinatorial argument. $\square$

Case: $k/2 < \ell < k$

Not so easy...

- To be good, first ℓ order isomorphic to last ℓ
- When overlap is $\ell \leq k/2$, first ℓ and last ℓ are distinct
- When overlap is $k/2 < \ell < k$, not distinct
- This makes the second case difficult

Case: $\ell = k - 1$

Lemma (A,D,P-C,S,Y)

For $\ell = k - 1$, there are only 2 good permutations, namely the monotone permutations.

Proof (outline)

For a given k and $\ell = k - 1$ assume that a permutation π is good and not monotone. We can apply contradiction to show that the permutation actually must not be good or must be monotone.

1. Case on not starting and not ending π with 1
2. Case on starting π with 1
3. Case on ending π with 1

BIG Conjecture

Interesting pattern

Using a program in SageMath, we came up with the following conjectures:

1. For $\ell = k - 2$ there are* $2(k + 2)$ good permutations
2. For $\ell = k - 3$ there are* $2(k^2 + 3k + 6)$ good permutations
3. For $\ell = k - 4$ there are* $2(k^3 + 3k^2 + 14k + 24)$ good permutations

   *=pending proof

BIG Conjecture

Given k and $\ell = k - r$, the number of good permutations is polynomial in k of degree $r - 1$.

How to prove conjectures?

Conjecture (A,D,P-C,S,Y)

Suppose π is a good permutation on [k] and $\ell = k - r$. Then the entries which overlap differ by at most r.

Lemma (A,D,P-C,S,Y)

If π is a good permutation, then π_r, the reverse of π is also a good permutation.

Connection to expectation

Main problem

What is the expected number of distinct permutation patterns contained in a random permutation on $[k]$?

Helper problems

Verifying three levels of order isomorphism:

1. Determining whether two random permutations are order isomorphic if they are disjoint. (Easy)
2. Determining whether two random permutations are order isomorphic if they overlap in consecutive positions. (Focus)
3. Determining whether two random permutations are order isomorphic if they overlap in non-consecutive positions. (Future work)

What is known?

Theorem (Downs, Godbole, Fokuoh, Fonseca)

If π and μ are two random permutations on $[k]$ which overlap consecutively, and $\ell \leq k/2$, then $\mathbb{P}(\pi \simeq \mu) \leq 3^k/k!$.

Remark

This bound isn't actually bad! However, it would be nice to apply some novel method to gain an improvement!

Connection to Good Permutations

Theorem (A,D,P-C,S,Y)

Given two permutations π_1 and π_2 that overlap

$$\begin{array}{ll} \pi_1 : & \overbrace{a_1\ a_2\ \cdots\ a_\ell}^{\rho_1}\ \cdots\ \overbrace{a_{k-\ell}\ \cdots\ a_k}^{\rho_2} \\ \pi_2 : & \underbrace{b_1\ b_2\ \cdots\ b_\ell}_{\rho_1}\ \cdots\ b_{k-\ell}\ \cdots\ b_k \end{array}$$

Then $\rho_1 \simeq \rho_2$ if and only if π_1 and π_2 are consistent.

Proof

($\Rightarrow$) *Algorithmic Approach* Create a sequence of symbols which designates two consistent permutations.
($\Leftarrow$) *Proof by Contradiction* Suppose that $\pi_1 \simeq \pi_2$. Assume that ρ_1 and ρ_2 are not order isomorphic.

Recap

What have we done?

1. We have defined and attempted to enumerate good permutations.
2. We have connected good permutations to the problem of determining the expected number of distinct permutation patterns in a random permutation.
3. Work has been done, but the purpose of the "polynomial" method is improvement.
4. We want to apply this to the non-consecutive overlap case.

The End

Special thanks to the JMM organizers of this special undergraduate session and to the organizers of JMM.

Special thanks to ETSU for hosting the (virtual) REU, to the NSF for funding the experience, and to our advisor Prof. Anant Godbole.