Introducing Three Best Known Binary Goppa Codes

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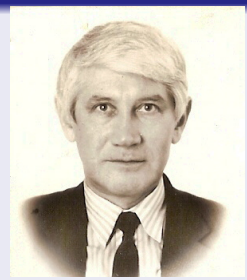
Introduction

History

Binary Goppa codes were introduced in 1970 by Russian mathematician Valery Denisovich Goppa (born 1939–) who discovered the relation between algebraic geometry and codes.



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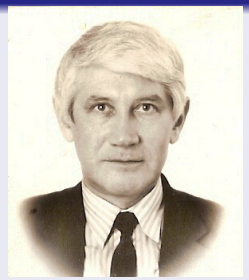

Modern-day Applications

1. Post-quantum cryptosystems (McEliece cryptosystem)

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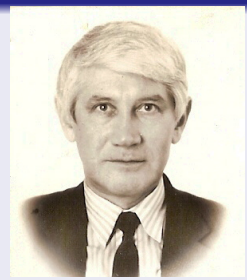

Modern-day Applications

1. Post-quantum cryptosystems (McEliece cryptosystem)
2. Telecommunication

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- The dimension of C is $\dim(C) = \log_2 |C|$.
- The minimum distance of C is

$$d(C) = \min\{d_H(x, y) : x, y \in C, x \neq y\},$$

where

$$d_H(x, y) := \big|\{i \mid x_i \neq y_i\}\big|.$$

Background

Definition

Let p be a prime and let $q = p^m$. Let $L = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a subset of $\mathbb{F}_q$. Let $g(x)$ be a polynomial such that $g(\alpha_i) \neq 0$, for $\alpha_i \in L$. The p–ary Goppa code determined by L and g is defined as

$$C(L, g) := \left\{ (c_1, c_2, \ldots, c_n) \in \mathbb{F}_p^n \;\middle|\; \sum_{i=1}^{n} \frac{c_i}{x - \alpha_i} \equiv 0 \mod g(x) \right\}.$$

Background

Definition

Given a linear code C with length n, let A_w denote the number of codewords whose weight equals w. Then, the vector $[A_0, A_1, ..., A_n]$ is called the weight enumerator of C. The weight enumerator polynomial of C is defined by

$$W(C; x, y) = \sum_{w=0}^{n} A_w x^w y^{n-w}.$$

The lowest positive weight w such that $A_w \neq 0$ is the minimum distance of the code.

Background

Definition

Given $m, s \in \mathbb{Z}^+$, the trace function $\mathrm{Tr}_{m,s}(X)$ is the mapping $\mathrm{Tr}_{m,s} : \mathbb{F}_{p^{ms}} \mapsto \mathbb{F}_{p^s}$ given by

$$\mathrm{Tr}_{m,s}(X) = \sum_{i=0}^{m-1} X^{p^{is}} = X + X^{p^s} + X^{p^{2s}} + \cdots + X^{p^{(m-1)s}}.$$

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Definition

Given $m, s \in \mathbb{Z}^+$, the norm function $\mathrm{Norm}_{m,s}(X)$ is the mapping $\mathrm{Norm}_{m,s} : \mathbb{F}_{p^{ms}} \mapsto \mathbb{F}_{p^s}$ given by

$$\mathrm{Norm}_{m,s}(X) = \prod_{i=0}^{m-1} X^{p^{is}} = X^{1+p^s+p^{2s}+\cdots+p^{(m-1)s}}.$$

Background

Previous Research

P. Verón improved the dimension bound for binary Goppa codes whose Goppa polynomial is given by

$$g(x) = a(x)\,\mathrm{Tr}(b(x)),$$

where $a(x)$ and $b(x)$ are polynomials over $\mathrm{F}_{p^{ms}}[X]$.

Recently, other improvements have been made to the dimension and minimum distance bounds of q-ary Norm Goppa codes ($q = p^s$), i.e. Goppa codes over $\mathbb{F}_q$ where the Goppa polynomial is $\mathrm{N}_{\mathbb{F}_{q^m}\setminus\mathbb{F}_q}(b(x)) = b(x)^{1+q+q^2+\cdots q^{m-1}}$.

Our Goppa polynomials are related to these classes codes because $x^{16} + x = \mathrm{Tr}_{\mathbb{F}_{2^8}\setminus\mathbb{F}_{2^4}}(x)$, $x^{15} + 1 = \dfrac{\mathrm{Tr}_{\mathbb{F}_{2^8}\setminus\mathbb{F}_{2^4}}(x)}{x}$, and $x^{17} + 1 = \mathrm{N}_{\mathbb{F}_{2^8}\setminus\mathbb{F}_{2^4}}(x) + 1$.

Code Constructions

Construction of [239,21,103] code

We computed that the binary Goppa code $C\left(L,(x^{17}+1)^6\right)$ is a $[239,21,103]$ code. This linear code has a higher minimum distance than the current best known $[239,21,98]$ binary code.

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Weight Enumerator Polynomial

$$\begin{aligned} x^{239} &+ 62244x^{136}y^{103} + 81396x^{135}y^{104} + 190519x^{128}y^{111} + 217736x^{127}y^{112} \\ &+ 496680x^{120}y^{119} + 496680x^{119}y^{120} + 217736x^{112}y^{127} + 190519x^{111}y^{128} \\ &+ 81396x^{104}y^{135} + 62244x^{103}y^{136} + y^{239}. \end{aligned}$$

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Construction of Derived Codes

By puncturing the $[239,21,103]$ code 12 times we get best known codes with the following parameters:

$$[238,21,102],\ [237,21,101],\ [236,21,100],\ [235,21,99],$$
$$[234,21,98],\ [233,21,97],\ [232,21,96],\ [231,21,95],$$
$$[230,21,94],\ [229,21,93],\ [228,21,92],\ \text{and}\ [227,21,91].$$

Code Constructions

Construction of [240,21,104] code:

The binary Goppa code $C\left(L,(x^{16}+x)^6\right)$ is a $[240,21,104]$ code. This linear code has a higher minimum distance than the current best known $[240,21,98]$ binary code.

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Construction of [240,21,104] code:

The binary Goppa code $C\left(L,(x^{16}+x)^6\right)$ is a $[240,21,104]$ code. This linear code has a higher minimum distance than the current best known $[240,21,98]$ binary code.

Weight Enumerator Polynomial

$$x^{240}+143640x^{136}y^{104}+408255x^{128}y^{112}+993360x^{120}y^{120}$$
$$+408255x^{112}y^{128}+143640x^{104}y^{136}+y^{240}.$$

Code Constructions

Construction of [240,21,104] code:

The binary Goppa code $C\left(L,(x^{16}+x)^6\right)$ is a $[240,21,104]$ code. This linear code has a higher minimum distance than the current best known $[240,21,98]$ binary code.

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Construction of Derived Codes

By shortening our $[240,21,104]$ code we get a $[239,20,104]$ code. By puncturing this one 7 times we get codes with the following parameters:

$[238,20,103]$, $[237,20,102]$, $[236,20,101]$, $[235,20,100]$, $[234,20,99]$, $[233,20,98]$, and $[232,20,97]$.

Code Constructions

Construction of [241,21,104] binary code:

The binary Goppa code $C\left(L,(x^{15}+1)^6\right)$ is a $[241,21,104]$ code. This linear code has a higher minimum distance than the current best known $[241,21,99]$ binary code.

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Construction of [241,21,104] binary code:

The binary Goppa code $C\left(L,(x^{15}+1)^6\right)$ is a $[241,21,104]$ code. This linear code has a higher minimum distance than the current best known $[241,21,99]$ binary code.

Weight Enumerator Polynomial

$$x^{240}y + 143640x^{136}y^{105} + 408255x^{128}y^{113} + 993360x^{120}y^{121} + 408255x^{112}y^{129} + 143640x^{104}y^{137} + y^{241}.$$

Code Constructions

Construction of [241,21,104] binary code:

The binary Goppa code $C\left(L,(x^{15}+1)^6\right)$ is a $[241,21,104]$ code. This linear code has a higher minimum distance than the current best known $[241,21,99]$ binary code.

Weight Enumerator Polynomial

$$x^{240}y + 143640x^{136}y^{105} + 408255x^{128}y^{113} + 993360x^{120}y^{121} + 408255x^{112}y^{129} + 143640x^{104}y^{137} + y^{241}.$$

Construction of Derived Codes

By extending the $[241,21,104]$ code to further lengths we get codes with following parameters:

$$[242,21,104],\ [243,21,104],\ [245,21,104],$$
$$[246,21,104],\ \text{and}\ [247,21,104].$$

Other Best Known Codes

More Code Constructions!

We are very grateful to M. Grassl for pointing out the following two constructions of new best known binary codes derived from the $[240, 21, 104]$ binary Goppa code.

With the technique from [3] it is actually possible to puncture the code at suitably chosen positions to obtain best known binary codes of parameters $[208, 21, 81]$, $[210, 21, 82]$, $[213, 21, 84]$, $[215, 21, 85]$, $[218, 21, 87]$, $[220, 21, 88]$, $[223, 21, 90]$, $[226, 21, 92]$, $[229, 21, 94]$, and $[229, 21, 94]$, but the very positions depend on the choice of the ordering of the elements of $\mathbb{F}_{256}$ when constructing the Goppa code in first place.

Applying Construction X [4] to the $[240, 21, 104]$ binary Goppa code, we can also find a best known $[249, 21, 106]$ binary code and a best known $[254, 22, 106]$ binary code.

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