2021 Gulf Coast Undergraduate Research Symposium

Mathematics Abstract Book

Abstracts are listed in the order of presentation. If an abstract does not appear, it is because the student has requested it not be published.

Invariant Geometric Structures on Complex Almost Abelian Groups

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Abstract
An almost Abelian group is a non-Abelian Lie group with a codimension 1 Abelian subgroup. This project investigates invariant Hermitian and Kähler structures on complex almost Abelian groups. In doing so, we find explicit formulas for the left and right Haar measures, the modular function, and left and right generator vector fields on simply connected complex almost Abelian groups. From the generator fields, we obtain invariant vector and tensor field frames, allowing us to find an explicit form for all invariant tensor fields. Namely, all such invariant tensor fields have constant coefficients in the invariant frame. From this, we classify all invariant Hermitian forms on complex simply connected almost Abelian groups, and we prove the nonexistence of invariant Kähler forms on all such groups. Via constructions involving the pullback of the quotient map, we extend the explicit description of invariant Hermitian structures and the nonexistence of Kähler structures to quotients of complex simply connected almost Abelian groups.

This work was done as part of the University of California, Santa Barbara Mathematics Summer Research Program for Undergraduates and was supported by NSF REU Grant DMS 1850663.

Invariant Geometric Structures on Complex Almost Abelian Groups 1.

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Abstract

An almost Abelian group is a non-Abelian Lie group with a codimension 1 Abelian subgroup. This project investigates invariant Hermitian and Kähler structures on complex almost Abelian groups. In doing so, we find explicit formulas for the left and right Haar measures, the modular function, and left and right generator vector fields on simply connected complex almost Abelian groups. From the generator fields, we obtain invariant vector and tensor field frames, allowing us to find an explicit form for all invariant tensor fields. Namely, all such invariant tensor fields have constant coefficients in the invariant frame. From this, we classify all invariant Hermitian forms on complex simply connected almost Abelian groups, and we prove the nonexistence of invariant Kähler forms on all such groups. Via constructions involving the pullback of the quotient map, we extend the explicit description of invariant Hermitian structures and the nonexistence of Kähler structures to quotients of complex simply connected almost Abelian groups.

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Topological Properties of Almost Abelian Lie Groups

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Abstract
Topology studies certain properties of geometric objects and their behavior under continuous deformations. Some basic properties that are studied include compactness and connectedness. More complicated questions may also be probed: are there "holes" in a space? How can we characterize "holes" in a space? The tools of algebraic topology help us answer these questions. In particular, every shape has an associated *homotopy type* that encodes some abstract information about any holes it contains.

A *Lie group* is a mathematical object that possesses both algebraic and geometric properties. Formally, a Lie group is a group that is also a smooth manifold, such that the group operation and inversion map are both smooth functions on the Lie group. A group is an algebraic object that resembles the more familiar vector space, albeit with slightly less structure. Every group has an operation analogous to how vector spaces come equipped with vector addition. If this group operation is commutative, we say that the group is *Abelian*. On the other hand, a smooth manifold is a geometric object that locally "resembles" flat (Euclidean) space, combined with a *smooth structure* which allows us to do calculus on the manifold. These objects may seem contrived, but they find application in particle physics.

Since Lie groups are smooth manifolds which are geometric objects, we may study the topological properties of Lie groups. Our work focuses on the topological properties of certain a certain family of Lie groups. If a Lie group is non-Abelian but has a codimension 1 Abelian subgroup, we say that the Lie group is almost Abelian. We show that all discrete subgroups of complex simply connected almost Abelian groups are finitely generated. We then prove that no complex connected almost Abelian group is compact and give conditions for the compactness of connected subgroups of such groups. Towards studying the homotopy type of complex connected almost Abelian groups, we investigate the maximal compact subgroups of such groups.

As Lie groups are both algebraic and geometric objects, it is a recurring theme in Lie theory to use algebraic techniques to study the geometry of Lie groups. In our work, the topology of connected almost Abelian Lie groups is studied by expressing each connected almost Abelian Lie group as a quotient of its universal covering group by a discrete normal subgroup. This proves to be a fertile approach and allows us to probe the various topological questions we ask of almost Abelian Lie groups.

Acknowledgement – This work was supported by NSF grant DMS-1850663.

Automorphisms of the fine curve graph

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Abstract

The curve graph of a surface is a graph whose vertices are isotopy classes of simple, closed, essential curves on the surface, and there exists an edge between two vertices if the two isotopy classes are disjoint.

Ivanov [1] showed that the automorphism group of the curve graph is isomorphic to the extended mapping class group, the group of isotopy classes of all homeomorphisms of a surface. We extend Ivanov's work by considering the fine curve graph of a surface (defined by Bowden, Hensel, and Webb [2]), where vertices of the graph are all curves on the surface, and there exists an edge between two vertices if the curves are disjoint.

Similar to Ivanov's work, we show that the automorphism group of the fine curve graph is isomorphic to the extended mapping class group. This is joint work with Dr. Dan Margalit and Dr. Yvon Verberne. This project stems from work completed at the Georgia Tech REU.

References:

[1] J. Bowden, S. Hensel, and R. Webb. Quasi-morphisms on surface diffeomorphism groups. arXiv:1909.07164, 2019. To appear in J. Amer. Math. Soc.

[2] Nikolai V. Ivanov. Automorphism of complexes of curves and of Teichmüller spaces. Internat. Math. Res. Notices, (14):651–666, 1997.

Acknowledgement: This work was supported by NSF Grant DMS-181843.

Braid Indices of 1-Bridge Braids

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Abstract

Many interesting 3-dimensional spaces arise as the complement of knots embedded into three-dimensional spaces. Thus, knots are an important tool in studying low-dimensional topology, the study of shapes and spaces of dimension at most four. An important and well-studied family of knots to consider are 1-bridge braids. To help us better understand these knots, we study a knot invariant called the braid index. Knot invariants are a crucial concept in knot theory because they can sometimes tell us when two knots are different. Employing Markov's Theorem and a result of Morton and Franks-Williams, we compute the braid index for any 1-bridge braid, K(w,b,t), in terms of the number of strands w, the bridge number b, and the number of twists t. Future work will look at the insights this yields about certain 3-dimensional manifolds.

Acknowledgement- This joint work began as part of the 2021 iteration of the Georgia Tech Research Experience for Undergraduates program. We gratefully acknowledge support from NSF grants DMS-1552285(SK, ML), DMS-1745583 (DG, SK, VN, IT, LW), and DMS-1902729 (ML).

Enumerating Polyominoes on the Torus and Other Finite Surfaces

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Abstract

Since their popularization in the mid-20th century, significant attention has been directed to the counting of polyominoes on an infinite plane. A polyomino is a shape consisting of unit squares that are connected by their edges. In our research, we pursue the less examined problem of counting polyominoes on finite surfaces with different dimensions, such as the torus and Möbius strip. Specifically, we examine the fixed polyominoes, where different orientations of a polyomino are considered distinct. We first perform by-hand enumeration for surfaces of smaller dimensions to learn more about the properties of polyominoes on each surface. Utilizing existing algorithms as a foundation, we then employ computational methods to enumerate and count polyominoes for larger cases. Currently, we have counted up to the 19-ominoes on the 6×6 torus, cylinder, and finite grid. We have also found examples of polyominoes that exist on the torus but not on the finite grid and analyzed the conditions for this to occur.

Acknowledgement - This presentation will summarize the results of the CC-REU NSF summer REU experience (DMS-2050692) where these questions were explored.

Tamagawa Products for Elliptic Curves Over Number Fields

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Abstract

In recent work, Griffin, Ono, and Tsai constructs an L-series to prove that the proportion of short Weierstrass elliptic curves over $\mathbb{Q}$ with trivial Tamagawa product is $0.5054\dots$ and that the average Tamagawa product is $1.8183\dots$. Following their work, we generalize their L-series over arbitrary number fields K to be

$$L_{\text{Tam}}(K;s) := \sum_{m=1}^{\infty} \frac{P_{\text{Tam}}(K;m)}{m^s},$$

where $P_{\text{Tam}}(K;m)$ is the proportion of short Weierstrass elliptic curves over K with Tamagawa product m. We then construct Markov chains to compute the exact values of $P_{\text{Tam}}(K;m)$ for all number fields K and positive integers m. As a corollary, we also compute the average Tamagawa product $L_{\text{Tam}}(K;-1)$. We then use these results to uniformly bound $P_{\text{Tam}}(K;1)$ and $L_{\text{Tam}}(K,-1)$ in terms of the degree of K. Finally, we show that there exist sequences of K for which $P_{\text{Tam}}(K;1)$ tends to 0 and $L_{\text{Tam}}(K;-1)$ to ∞, as well as sequences of K for which $P_{\text{Tam}}(K;1)$ and $L_{\text{Tam}}(K;-1)$ tend to 1.

Acknowledgement. We are grateful for the generous support of the National Science Foundation (Grants DMS 2002265 and DMS 205118), National Security Agency (Grant H98230-21-1-0059), the Thomas Jefferson Fund at the University of Virginia, and the Templeton World Charity Foundation. The authors would like to thank Ken Ono and Wei-Lun Tsai for many helpful suggestions and conversations.

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An Unconditional Explicit Bound on the Error Term in the Sato-Tate Conjecture

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Abstract

Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ be a holomorphic cuspidal newform with even integral weight $k \geq 2$, level N, trivial nebentypus, and no complex multiplication (CM). For all primes p, we may define $\theta_p \in [0,\pi]$ such that $a_f(p) = 2p^{(k-1)/2}\cos\theta_p$. The Sato-Tate Conjecture states that the angles θ_p are equidistributed with respect to the probability measure $\mu_{\mathrm{ST}}(I) = \frac{2}{\pi}\int_I \sin^2\theta \, d\theta$, where $I \subseteq [0,\pi]$. Using recent results on the automorphy of symmetric-power L-functions due to Newton and Thorne, we construct the first unconditional explicit bound on the error term in the Sato-Tate Conjecture, which applies when N is squarefree as well as when f corresponds to an elliptic curve with arbitrary conductor. In particular, if $\pi_{f,I}(x) := \#\{p \leq x : p \nmid N, \theta_p \in I\}$, and $\pi(x) := \#\{p \leq x\}$, we have the following bound for $x \geq 3$:

$$\left|\frac{\pi_{f,I}(x)}{\pi(x)} - \mu_{\mathrm{ST}}(I)\right| \leq 58.1\frac{\log((k-1)N\log x)}{\sqrt{\log x}}.$$

As an application, we give an explicit bound for the number of primes up to x that violate the Atkin-Serre Conjecture for f.

Acknowledgement – This work was supported by the National Science Foundation [DMS 2002265, DMS 205118]; National Security Agency [H98230-21-1-0059]; the Thomas Jefferson Fund at the University of Virginia; and the Templeton World Charity Foundation.

Alternative Expressions And Study Of The Odd Integral Values Of The Riemann Zeta Function

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Abstract

The Riemann Zeta Function (defined by $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}$, Re(s)>1), introduced by Leonhard Euler back in the early 18th century and made popular by the German Mathematician Bernhard Riemann in his landmark 1859 paper titled “On the number of primes less than a given quantity”, has received significant attention from a large number of mathematicians over the centuries. It appears very frequently in numerous areas of mathematics, like Number Theory(especially in the study of distribution of prime numbers), Complex Analysis and even in Physics, like during the study of Casimir Effect. It is often dubbed as the “most important function in mathematics”. Though we have a closed form expression for the even positive integral values(2,4,6,…) of the function and know about their algebraic nature, much less is known about the properties of the odd integral values(3,5,7,…). This is one of the most active fields of research in Number Theory(along with the famous Riemann’s Hypothesis) and has been addressed from time to time by famous mathematicians like Srinivasa Ramanujan, Bruce Berndt, Roger Apery(who proved the irrational nature of ζ(3), the Apery’s constant in 1978), Yuri Nesterenko, Wadim Zudilin, Frits Beukers etc. On the other hand, there are numerous beautiful formulas for the function(apart from the standard infinite sum representation)involving other expressions, known as alternate representations of the same. In this talk, we will focus mainly on 2 aspects of the function – its representation using sums, products, integrals, Special functions and its expression, behavior at odd positive integral values. We will see some new representations of the Riemann Zeta Function involving Gamma function, Hyperbolic functions, Trigonometric Functions and Infinite Summation, Product. We would also discuss an alternative expression of a related interesting result of Ramanujan, involving the Digamma Function. Overall, the talk will involve topics from both Number Theory and Analysis.

Lower Bounds on the Canonical Height of Elliptic Curves of Non-Torsion Points on Mordell Curves

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Abstract

The theory of elliptic curves is one with many unbroken conjectures: for instance, the Millennium Prize BSD Conjecture, Goldfeld's 1979 conjecture concerning the asymptotic frequency of a fixed rank within a quadratic twist family of an elliptic curve.

This is what takes me to my research project this past summer, where we uncover two more such conjectures. My project was inspired by Le Boudec's 2016 paper "Height of Rational Points on Quadratic Twists of a Given Elliptic Curve", where he studies the family of quadratic twists of the congruent number curve of the form $dy^2 = x^3 - x$, where d is a positive square-free integer. In this paper, for each such curve, he considers the minimal value of the canonical height of a rational point on the curve and asks if there is a lower bound for a natural density 1 subset of these values in terms of d. He provides a positive answer to this question: $d^{5/8-\varepsilon}$ where $\varepsilon > 0$.

A critical step in Le Boudec's proof is the quantitative arithmetic of projective varieties, which reduces his assertion to counting integer points on a curve in a bounded region of affine space. We utilize the same tactics in this research.

Hence, in this project, we ask if such a bound can be achieved for the following families of curves: the family $y^2 = x^3 + d$ where $d \neq 0$ is a sixth-power free integer. We obtain a bound of $d^{2/9-\varepsilon}$ in this case.

The bounds stated by Le Boudec and I are far from optimal in quite an extreme sense. In Le Boudec's case, he conjectures by the analogy of quadratic number fields and quadratic twists of elliptic curves — formalized by Christophe Delaunay — that the lower bound can be optimized to $e^{d^{1/2}-\varepsilon}$ where $\varepsilon > 0$. And by the extended version of Goldfeld's 1979 Conjecture and the fact that Delaunay's analogy is between number fields and elliptic curves in general, I further conjecture that in both of my cases, we can also achieve a lower bound of $e^{d^{1/6}-\varepsilon}$ where $\varepsilon > 0$. From here, one can easily see the huge rift between conjecture and theory that remains to be resolved. The quantitative arithmetic of projective varieties seems well-positioned to make great strides here.

Acknowledgment — This summer research was supported by a grant funded by the Rabi Scholars Program of Columbia University. I would like to thank Dr. Chao Li of the Columbia University Department of Mathematics for guiding me this past summer with numerous useful heuristics, both in research and my future undergraduate studies. I am especially grateful to both parties for their gracious handling of the remote research required during the COVID-19 pandemic. Their cooperation with me during these awkward times has been exceptional.

A Theory for Locus Ellipticity of Poncelet Triangle Centers

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Abstract
Given a pair of ellipses, one inside the other, one can build a trajectory starting at some point P on the outer ellipse, tracing a tangent from there to the inner ellipse, intersecting this tangent with the outer ellipse to get a new point and repeat from there. Poncelet's Porism states that if this trajectory closes for some initial point P, then it will always close for every initial point P on the outer ellipse, always with the same number of steps. When this happens, we get an infinite family of polygons, all with the same number of sides, which inscribe the outer ellipse and circumscribe the inner ellipse. If the number of sides is *n*, we call these polygons *n-periodics*.

When studying families of Poncelet 3-periodics, many invariants like constant perimeter or area show up, especially in particular pairs of ellipses such as when one of them is a circle, or when the ellipses are confocal (in this case, the Poncelet 3-periodics are exactly the 3-periodic orbits of an elliptic billiard). Moreover, when considering the loci of triangle centers such as the barycenter, circumcenter, or orthocenter, one observes that they are very often ellipses themselves.

In our work, we prove that when a triangle center is a fixed affine combination of the barycenter and circumcenter, its locus is an ellipse over any Poncelet family of 3-periodics. Moreover, we extend this result for the confocal families (elliptic billiard) and the "incircle families" and prove that over 40 major triangle centers have elliptic loci over such families.

A Theory for Locus Ellipticity of Poncelet 3-Periodic Centers - https://arxiv.org/abs/2106.00715

%-Immanants and Kazhdan-Lusztig Immanants 1.

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Abstract

In this paper, we investigate the relationship between Kazhdan-Lusztig immanants, which were introduced in Rhoades-Skandera (2006), and %-immanants, which were introduced in Chepuri-Sherman-Bennett (2021). As noted by Skandera (2008), Kazhdan-Lusztig immanants are important dual canonical basis elements of the coordinate ring of GL_n (C). Our main result is a classification of when a 321-avoiding Kazhdan-Lusztig immanant can be written as a linear combination of %-immanants. This result uses a formula in Rhoades-Skandera (2005) to compute 321-avoiding Kazhdan-Lusztig immanants. We also partially extend our classification to general Kazhdan-Lusztig immanants: we obtain a necessary condition for a Kazhdan-Lusztig immanant to be a linear combination of %-immanants and a classification for a Kazhdan-Lusztig immanant to be written as a sum of at most two %-immanants. Finally, we conjecture an explicit formula for computing the Kazhdan-Lusztig immanants coming from a 1324-, 32154-, 21543-avoiding permutation, and using this conjectural formula, we derive expressions for 1324-, 24153-, 31524-, 32154-, 21543-, 231564-, 312645-, 426153-avoiding Kazhdan-Lusztig immanants as a sum of %-immanants.

This project was partially supported by RTG grant NSF/DMS-1148634, DMS-1949896, and the Office of Undergraduate Research at Washington University in St. Louis. It was supervised as part of the University of Minnesota School of Mathematics Summer 2021 REU program.

The authors would like to thank Professor Pavlo Pylyavskyy for introducing the problem and offering helpful directions for research and Sylvester Zhang for their mentorship and helpful comments on the paper. In addition, the authors would like to thank Swapnil Garg and Brian Sun for their algorithmic and coding support for this project.

Friezes from Dissections over $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$

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Abstract

A frieze on a polygon is a map from the diagonals of the polygon to an integral domain which respects the Ptolemy relation. In other words, it assigns a weight to every arc of the polygon in such a way that the Ptolemy relation is satisfied. We can obtain a frieze on a polygon by dissecting the polygon into sub-polygons and let all the diagonals from dissection have weight 1. One example of a frieze on an n-gon is treating the n-gon as a regular polygon. Holm and Jørgensen recently studied friezes which are the result of pasting together these regular friezes. This was a generalization of work by Conway and Coxeter, who found that friezes over positive integers on an n-gon are in bijection with triangulations of an n-gon.

In particular, we are interested in *unitary* friezes on a polygon. A frieze is unitary if there exists a triangulation of the underlying polygon such that every arc in the triangulation has a weight in the frieze which is a unit in the respective integral domain. Conway and Coxeter found that all friezes on a polygon over positive integers are unitary. Gunawan and Schiffler showed that positive, integral friezes on cluster algebras of type $\widetilde{A}_{p,q}$ must be unitary by looking at friezes over an annulus.

Motivated by Holm and Jørgensen's study of friezes from dissections of a polygon into sub-polygons, we study friezes over the integral domain $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$. The ring $\mathbb{Z}[\sqrt{2}]$ corresponds to dissecting a polygon into triangles and (regular) quadrilaterals, and the ring $\mathbb{Z}[\sqrt{3}]$ corresponds to dissecting a polygon into triangles and (regular) hexagons. Symmetries for friezes arising dissections purely into quadrilaterals or hexagons were already studied by Lukas Andritsch. We explore the characterization of unitary friezes over these integral domains. We identify families of dissections that produce unitary and non-unitary friezes. Among them, we identify a family of dissections which give rise to a unitary frieze in $\mathbb{Z}[\sqrt{2}]$; their analogue in $\mathbb{Z}[\sqrt{3}]$ retain some of the nice properties but ultimately fail to give a unitary frieze. We conjecture that in $\mathbb{Z}[\sqrt{2}]$, a dissection into triangles and quadrilaterals produce a unitary frieze if and only if the dissection is a gluing of towers; and in $\mathbb{Z}[\sqrt{3}]$, a dissection into triangles and quadrilaterals produce a unitary frieze if and only if the dissection is a gluing of triangles. We prove this conjecture in the case of certain types of dissections. We also show the weights of a family of arcs from these shapes of towers correspond to nice, infinite continued fractions.

This project was partially supported by NSF RTG grant DMS-1148634 and NSF grant DMS-1949896.

On 2×2 tropical commuting matrices

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Abstract

This talk investigates the geometric properties of a special case of the two-sided system given by 2×2 tropical commuting constraints. Given a finite matrix $A \in \mathbb{R}^{2\times 2}$, we study the extremals of the tropical polyhedral cone generated by the entries of matrices B such that $A \otimes B = B \otimes A$ and proposes a criterion to test whether two 2×2 matrices commute in max linear algebra.

Acknowledgement – Many thanks to Dr. Ngoc M. Tran for guidance throughout this project. Also thanks to George D. Torres and Luyan Yu for very helpful insights in tropical polyhedral cones, and Yaoyang Liu and Amanda J. Priestley for comments on earlier stages of this project.

Induced Star-Saturated Graphs

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Abstract

Using the definition of Tennenhouse, a graph G is *induced H-saturated* if there exists no induced subgraph H in G, but for every edge $e \in \overline{G}$, $G + e$ has an induced subgraph H. Tennenhouse showed the existence of induced $K_{1,3}$-saturated graphs for $n \geq 12$. Inspired by his results for this star graph $K_{1,3}$, we show there exist induced $K_{1,3}$-saturated graphs on n vertices if and only if $n \geq 8$. Additionally, we construct arbitrarily large induced $K_{1,m}$-saturated graphs for $4 \leq m \leq 7$ by adopting techniques from Behrens, Erbes, Santana, Yager, and Yeager. Finally, for the double star $D_{2,2}$, we show via a constructive proof that there exists an induced $D_{2,2}$-saturated graph on n vertices if and only if $n \geq 12$.

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Linear Preservers of Eigenvalues Induced by the Two-Dimensional Ice Cream Cone

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Abstract
Given a real $n \times n$ matrix A and a closed convex cone $K \subseteq \mathbb{R}^n$, the *eigenvalue complementarity problem* generalizes the eigenvalue problem by finding $\lambda \in \mathbb{R}$ and nonzero $x \in \mathbb{R}^n$ such that $K \ni x \perp (A - \lambda I)x \in K^*$, where K^* denotes the dual cone of K. We say λ is a *Lorentz eigenvalue* of A when K is the *Lorentz cone* $\{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}: ||x||_2 \leq x_n\}$, also called the ice cream cone.

Linear maps on the space M_n of real $n \times n$ matrices and the space S_n of real symmetric $n \times n$ matrices preserving Lorentz eigenvalues are well studied in the literature when $n \geq 3$, assuming the map is of the form $A \mapsto PAQ$ or $A \mapsto PA^TQ$ for some matrices P and Q, which we call *standard linear maps*. Because the Lorentz cone for $n = 2$ is polyhedral, this case has been left out of the literature and turns out to be more complex than $n \geq 3$ in certain ways.

In our research, we fully characterize the standard linear maps that preserve the Lorentz spectrum on M_2 and S_2 and show that it is precisely these maps that preserve the *nature* of the Lorentz eigenvalues, i.e., whether they correspond to eigenvectors on the interior or boundary of the Lorentz cone. In the case of M_2, we show that different kinds of linear preservers are possible compared to M_n for $n \geq 3$. Moreover, we prove that all linear maps preserving the Lorentz spectrum on S_2 are standard and provide a criterion any nonstandard linear map preserving the Lorentz spectrum on M_2 must satisfy, though we conjecture that no such map exists.

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Preferential and *k*-Zone Parking Functions

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Abstract
Parking functions are vectors that describe the parking preferences of n cars that enter a one-way street containing n parking spots numbered 1 through n. A list of each car's preferences is also compiled into vectors in which we denote as $(a_1, \ldots, a_n)$, such that a_i is the parking preference for car i. The classical parking rule allows cars to enter the street one at a time going to their preferred parking spot and parking, if that space is unoccupied. If it is occupied, they then proceed down the one-way street and park in the first available parking spot. If all cars can park, we say the vector $(a_1, \ldots, a_n)$ is a parking function.

In our research, we introduce new variants of parking function rules with backward movement called k-Zone, preferential, and inverse preferential functions. We study the relationship between k-Zone parking functions and k-Naples parking functions and count the number of parking functions under these new parking rules which allows cars that find their preferred spot occupied to back up a certain parameter. One of our main results establishes that the set of non-increasing preference vectors are k-Naples if and only if they are k-Zone. For one of our findings we provide a table of values enumerating these new combinatorial objects in which we discover a unique relationship to the order of the alternating group A_{n+1}, numbers of Hamiltonian cycles on the complete graph, K_n, and the number of necklaces with n distinct beads for $n!$ bead permutations.

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Writing a Product of Prime Powers as a Sum of Recurrence Terms

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Abstract

We begin by extending the work of Pink and Ziegler (2018) to find all integer solutions to Diophantine equations such as $F_{n_1} + F_{n_1} + F_{n_1} = 3^{z_1}$ where F_n is the Fibonacci sequence. First, we find an explicit upper-bound on solutions (n_1, n_2, n_3, z_1) using lower bounds for linear forms in logarithms. Then, we introduce the LLL algorithm to significantly reduce this upper-bound, which allows us to perform a computational search for all integer solutions. In addition, we demonstrate how these techniques generalize to an algorithm for finding all integer solutions to equations consisting of a finite sum of binary recurrence terms equaling a product of prime powers. Finally, we also discuss how we can remove the binary condition and extend our family of Diophantine equations to consider recurrence sequences of higher order.

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